Star Formation

Q & A Session 07.07.2020

Protostar Formation & Evolution

A Simple Protostellar Evolution Model

Consider a protostar forming with a constant accretion rate $\dot{M}$. The accreting gas is fully molecular, arrives at free-fall, and radiates away a luminosity $L_{acc} = f_{acc}\, G\, M\, \dot{M} / R$ at the accretion shock, where M and R are the instantaneous protostellar mass and radius, and f_{acc} is a numerical constant of order unity. At the end of contraction the resulting star is fully ionized, all its deuterium has been burned to hydrogen, and it is in hydrostatic equilibrium.
The ionization potential of hydrogen is $\psi_I = 13.6$ eV per amu, the dissociation potential of molecular hydrogen is $\psi_M = 2.2$ eV per amu, and the energy released by deuterium burning is $\psi_D \approx 100$ eV per amu of total gas (not per amu of deuterium).

a)

First consider a low-mass protostar whose internal structure is well-described by an $n = 3/2$ polytrope. Compute the total energy of the star, including thermal energy, gravitational energy, and the chemical energies associated with ionization, dissociation, and deuterium burning.

The star is a polytrope, and for a polytrope of index n the gravitational energy is

$$\mathcal{W} = -\frac{3}{5-n}\frac{G M^2}{R}$$

According to the virial theorem the thermal energy is half the absolute value of the potential energy, so

$$\mathcal{T} = \frac{3}{2\,(5-n)}\frac{G M^2}{R}$$

Finally, the change in internal energy associated with dissociation, ionization, and deuterium burning is $(\psi_I + \psi_M - \psi_D)\, M$. Note the opposite signs: ψ_L and ψ_M are positive, meaning that the final state (ionized, atomic) is higher energy than the initial one, while ψ_D is negative, indicating that the final state (all the deuterium converted to He) is a lower energy state than the initial one. Putting this all together, the total energy of the star is

$$\mathcal{E} = -\frac{3}{2\,(5-n)}\frac{G M^2}{R} + (\psi_I + \psi_M - \psi_D)\, M$$

b)

Use your expression for the total energy to derive an evolution equation for the radius for a star. Assume the star is always on the Hayashi track, which for the purposes of this problem we will approximate as having a fixed effective temperature $T_H = 3500\,K$.

First we can compute the time rate of change of the star's energy,

$$\dot{\mathcal{E}} = -\frac{3}{2\,(5-n)}\,\frac{G\,M}{R}\left(M\,\frac{\dot{R}}{R} - 2\,\dot{M}\right) + (\psi_I + \psi_M - \psi_D)\,\dot{M}$$

Now consider conservation of energy. The star's luminosity L represents the rate of change of the energy "at infinity", i.e., the energy removed from the system. Since the total energy of the star plus infinity must remain constant, we require that $\dot{\mathcal{E}} + L = 0$. Writing down this condition and solving for $\dot{\mathcal{E}}$, we obtain

$$\dot{R} = 2\,R\,\frac{\dot{M}}{M} - \frac{2\,(5-n)}{3}\,\frac{R^2}{G\,M^2}\left[(\psi_I + \psi_M - \psi_D)\,\dot{M} + L\right]$$

It is convenient to divide by $\dot{M}$ in order to recast this as an equation for the evolution of R with M:

$$\frac{dR}{dM} = 2\,\frac{R}{M} - \frac{2\,(5-n)}{3}\,\frac{R^2}{G\,M^2}\left[(\psi_I + \psi_M - \psi_D) + \frac{L}{\dot{M}}\right]$$

Dividing by R/M on both sides we get

$$\frac{d\ln R}{d\ln M} = 2 - \frac{2\,(5-n)}{3}\,\frac{R}{G\,M}\left[(\psi_I + \psi_M - \psi_D) + \frac{L}{\dot{M}}\right]$$

Next, we must compute the total luminosity, which contains contributions from the star's intrinsic, internal luminosity, and from the accretion luminosity. Since the star is on the Hayashi track, we can compute the intrinsic luminosity by taking its effective temperature to be fixed at T_H. Thus the total luminosity is

$$L = L_{acc} + L_H = f_{acc}\,\frac{G\,M\,\dot{M}}{R} + 4\,\pi\,R^2\,\sigma\,T_H^4$$

which can be substituted in and we obtain:

$$\frac{d\ln R}{d\ln M} = 2 - \frac{2\,(5-n)}{3}\left[f_{acc} + \left(\frac{R}{G\,M}\right)\,(\psi_I + \psi_M - \psi_D + \frac{4\,\pi\,R^2\,\sigma\,T_H^4}{\dot{M}}\right]$$

c)

Numerically integrate your equation and plot the radius as a function of mass for $\dot{M} = 10^{-5}\,M_\odot\,\text{yr}^{-1}$ and $f_{acc} = 3/4$. As an initial condition, use $R = 2.5\,R_\odot$ and $M = 0.01\,M_\odot$, and stop the integration at a mass of $M = 1.0\,M_\odot$.

Plot the radius and luminosity as a function of mass; in the luminosity, include both the the accretion luminosity and the internal luminosity produced by the star.

In[173]:=
```
facc = 3/4;
Msun = 2 × 10^33;
Rsun = 6.96 × 10^10;
Lsun = 3.83 × 10^33;
eV = 1.6 × 10^-12;
amu = 1.66 × 10^-24;
year = 365.25 * 24 * 3600;
mdot = 10^-5 Msun/year;
TH = 3500;
ψI = 13.6 eV / amu;
ψM = 2.2 eV / amu;
ψD = 100 eV / amu;
σ = 5.67 × 10^-5; G = 6.67 × 10^-8;
dlnRdlnM[lnR_?NumberQ, lnM_?NumberQ, n_ : 1.5, facc_ : 0.75, Mdot_ : mdot] :=
 Module[{R = Exp[lnR], M = Exp[lnM]},
  (2 - (2 (5. - n))/3. * (facc + (R/(G M)) * (ψI + ψM - ψD + (4. π R^2 σ TH^4)/Mdot)))]
lum[R_, M_, n_ : 1.5, facc_ : 0.75, Mdot_ : mdot] := facc * G (M Mdot)/R + 4 π R^2 σ TH^4
```

Solving the DGL numerically gives $d \ln R / d \ln M$

In[188]:=
```
sol = NDSolveValue[{
   lnR'[lnM] == dlnRdlnM[lnR[lnM], lnM],
   lnR[Log[0.01 Msun]] == Log[2.5 Rsun]}, lnR, {lnM, Log[0.01 Msun], Log[1. Msun]}]
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Out[188]=
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InterpolatingFunction[ Domain: {{72.1, 76.7}} Output: scalar ]
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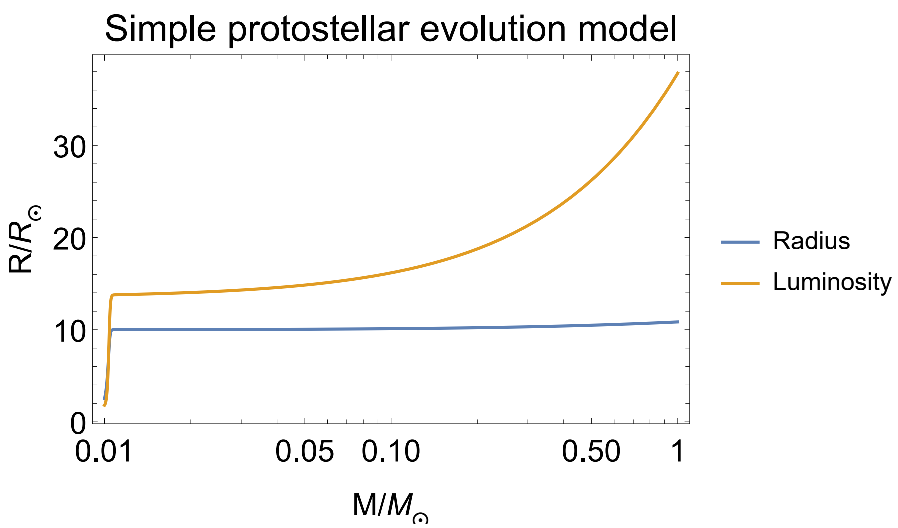
Out[∘]=



Note that the radius is too large by a factor of ~3 compared to more sophisticated models, mainly due to the incorrect assumption that all the accreted deuterium burns as quickly as it accretes. In reality the D luminosity should be significantly lower, because D burning lasts longer than accretion.

d)

Now consider two modifications we can make to allow the model to work for **massive protostars**. First, since massive stars are radiative, the polytropic index will be roughly $n = 3$ rather than

$n = 3/2$. Second, the surface temperature will in general be larger than the Hayashi limit, so take the luminosity to be $L = \max[L_H, L_\odot(M/M_\odot)^3]$, where $L_H = 4\pi R^2 \sigma T_H^4$ and R is the stellar radius. Modify your evolution equation for the radius to include these effects, and numerically integrate the modified equations up to $M = 50\, M_\odot$ for $\dot{M} = 10^{-4}\, M_\odot\, \mathrm{yr}^{-1}$ and $f_{acc} = 3/4$, using the same initial conditions as for the low mass case. Plot R and L versus M.

This problem can be solved using the same basic structure as the previous part. The derivative of radius with respect to mass now becomes

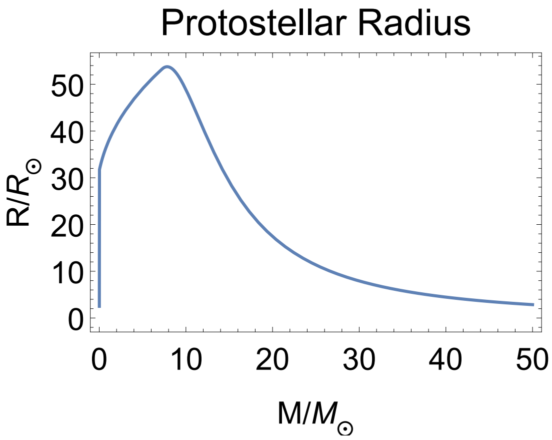
$$\frac{d \ln R}{d \ln M} = 2 - \frac{2\,(5-n)}{3}\left[f_{acc} + \left(\frac{R}{G M}\right)\left(\psi_I + \psi_M - \psi_D + \frac{\max\left[4\pi R^2 \sigma T_H^4,\ L_\odot\,(M/M_\odot)^3\right]}{\dot{M}}\right]\right.$$

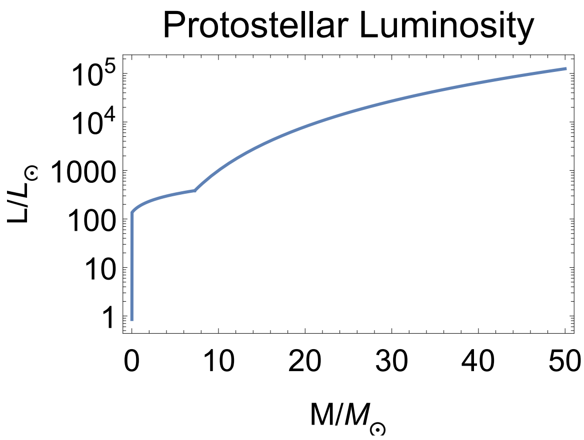
```
In[189]:= mdot2 = 10^-4 Msun/year;
dlnRdlnM2[lnR_?NumberQ, lnM_?NumberQ, n_ : 3, facc_ : 0.75, Mdot_ : mdot2] :=
 Module[{R = Exp[lnR], M = Exp[lnM]},
  (2 - (2 (5. - n))/3. * (facc + (R/(G M)) * (ψI + ψM - ψD + lumstar[R, M]/Mdot)))]
lumstar[R_, M_] := Max[4 π R^2 σ TH^4, Lsun (M/Msun)^3]

In[192]:= sol2 = NDSolveValue[{
    lnR'[lnM] == dlnRdlnM2[lnR[lnM], lnM], lnR[Log[0.01 Msun]] == Log[2.5 Rsun]},
   lnR, {lnM, Log[0.01 Msun], Log[50. Msun]}]

Out[192]= InterpolatingFunction[ Domain: {{72.1, 80.6}} Output: scalar ]
```

Out[156]=





e)

Compare your result to the fitting formula for the ZAMS radius of solar-metallicity stars as a function of M in Tout et al. (1996). Find the mass at which the massive star would join the main sequence. Your plots for R and L are only valid up to this mass, because this simple model does not include hydrogen burning.

$$L = \frac{\alpha M^{5.5} + \beta M^{11}}{\gamma + M^3 + \delta M^5 + \varepsilon M^7 + \zeta M^8 + \eta M^{9.5}}, \qquad (1)$$

where the coefficients α, β etc. are given in the first column of Table 1. For the radius

$$R = \frac{\theta M^{2.5} + \iota M^{6.5} + \kappa M^{11} + \lambda M^{19} + \mu M^{19.5}}{\nu + \xi M^2 + o M^{8.5} + M^{18.5} + \pi M^{19.5}}, \qquad (2)$$

In[157]:=

```
L[M_] := (0.39704170 M^5.5 + 8.52762600 M^11) /
  (0.00025546 + M^3 + 5.43288900 M^5 + 5.56357900 M^7 + 0.78866060 M^8 + 0.00586685 M^9.5)
R[M_] := (1 / (0.01077422 + 3.08 M^2 + 17.84778000 M^8.5 + M^18.5 + 0.00022582 M^19.5))
  (1.71535900 M^2.5 + 6.59778800 M^6.5 + 10.08855000 M^11 + 1.01249500 M^19 + 0.07490166 M^19.5)
```

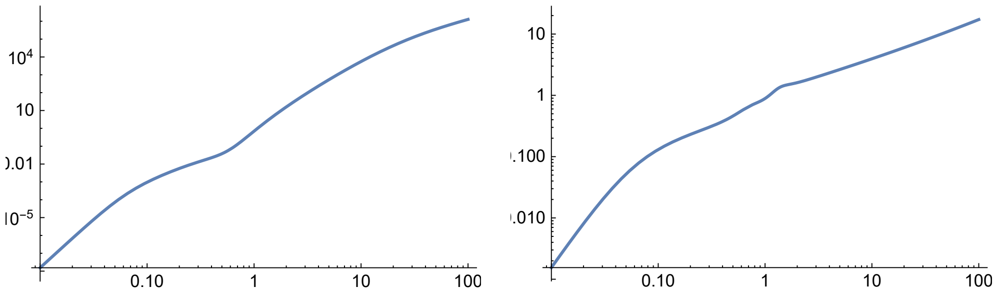
Table 1. Coefficients for equation (3).

	a	b	c	d	e
α	0.39704170	-0.32913574	0.34776688	0.37470851	0.09011915
β	8.52762600	-24.41225973	56.43597107	37.06152575	5.45624060
γ	0.00025546	-0.00123461	-0.00023246	0.00045519	0.00016176
δ	5.43288900	-8.62157806	13.44202049	14.51584135	3.39793084
ϵ	5.56357900	-10.32345224	19.44322980	18.97361347	4.16903097
ζ	0.78866060	-2.90870942	6.54713531	4.05606657	0.53287322
η	0.00586685	-0.01704237	0.03872348	0.02570041	0.00383376

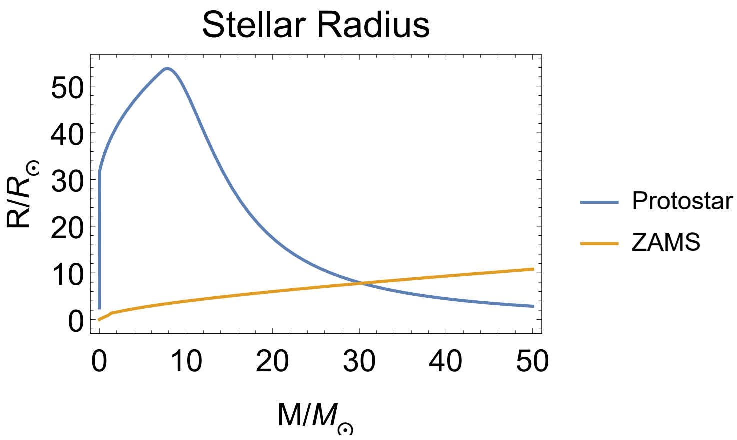
Table 2. Coefficients for equation (4).

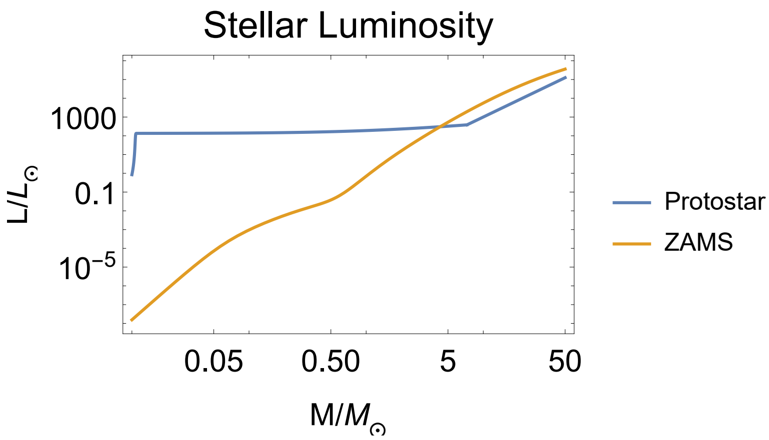
	a'	b'	c'	d'	e'
θ	1.71535900	0.62246212	-0.92557761	-1.16996966	-0.30631491
ι	6.59778800	-0.42450044	-12.13339427	-10.73509484	-2.51487077
κ	10.08855000	-7.11727086	-31.67119479	-24.24848322	-5.33608972
λ	1.01249500	0.32699690	-0.00923418	-0.03876858	-0.00412750
μ	0.07490166	0.02410413	0.07233664	0.03040467	0.00197741
ν	0.01077422	0	0	0	0
ξ	3.08223400	0.94472050	-2.15200882	-2.49219496	-0.63848738
o	17.84778000	-7.45345690	-48.96066856	-40.05386135	-9.09331816
π	0.00022582	-0.00186899	0.00388783	0.00142402	-0.00007671

Out[172]=



Out[170]=


Stellar Radius
R/R⊙
0
10
20
30
40
50
M/M⊙
Protostar
ZAMS


Stellar Luminosity
L/L⊙
1000
0.1
10⁻⁵
0.05
0.50
5
50
M/M⊙
Protostar
ZAMS